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COMMENT

Comment on 'Resolving isospectral 'drums' by counting nodal domains'

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Abstract

In Gnutzmann *et al* (2005 *J. Phys. A: Math. Gen.* **38** 8921–33) the authors studied the 4-parameter family of isospectral flat 4-tori $T^{\pm}(a, b, c, d)$ discovered by Conway and Sloane. With a particular method of counting nodal domains they were able to distinguish these tori (numerically) by computing the corresponding nodal sequences relative to a few explicit tuples (a, b, c, d). In this note we confirm the expectation expressed in Gnutzmann *et al* (2005 *J. Phys. A: Math. Gen.* **38** 8921–33) by proving analytically that their nodal count distinguishes any 4-tuple of distinct positive real numbers.

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1. Introduction

In 1964 J Milnor [4] constructed two 16-dimensional non-isometric flat tori with the same spectrum for the Laplace–Beltrami operator on forms of every degree, and thus produced the first example of non-isometric *isospectral* manifolds. Since then many examples of such manifolds (see for example [1, 5] and [2]) have been found and studied.

While it remains unclear to what extent the spectrum of the Laplace–Beltrami operator determines the geometry of the underlying manifold, these examples show that the spectrum does not contain enough information to determine the manifold and its metric uniquely. It has been proposed recently that the *nodal count*, i.e. the number of nodal domains of the eigenfunctions of the Laplace–Beltrami operator, might provide the missing information, such that the spectrum and nodal count together should yield isometry. Indeed, in [3] the authors used the 4-parameter family of isospectral flat 4-tori $T^{\pm}(a, b, c, d)$ constructed by Conway and Sloane [2] to show how the isospectrality can be 'resolved' using nodal domains. By counting the latter in a very special way, and then arranging the result in a so-called 'nodal sequence' they were able to exhibit that these nodal sequences for the tori belonging to four carefully chosen tuples (a, b, c, d) are different, if numerically.

In this work we shall give an alternative and analytic way to show that the nodal sequence defined in [3] distinguishes every pair of isospectral tori in this family if all four parameters are distinct; if at least two of them are equal then the corresponding tori are isometric (cf [2], p 94, Remark 2).

Theorem. *For any choice of distinct positive numbers $a, b, c, d \in \mathbb{R}_+$ the nodal sequences of the tori $T^+(a, b, c, d)$ and $T^-(a, b, c, d)$ are distinct.*

The paper is structured as follows: in section 2 we introduce some facts on flat *n*-tori, their spectra and eigenfunctions, and define the notions of 'nodal domain' and 'nodal sequence'. We then proceed to introduce the isospectral flat tori of Conway and Sloane in section 3, and eventually prove the theorem stated above.

2. Nodal sequences of flat tori

Let $v_1, \ldots, v_n \in \mathbb{R}^n$ denote linearly independent vectors and

$$\Gamma := \mathrm{span}_{\mathbb{Z}} \{v_1, \ldots, v_n\}$$

the lattice generated by these vectors. The flat torus given by the lattice Γ is

$$T := \mathbb{R}^n / A\mathbb{Z}^n,$$

where the columns of the $(n \times n)$-matrix A consist of the vectors v_i:

$$A = [v_1, \ldots, v_n].$$

The *Gram matrix* of T is defined by $G := A^\top A$ and $Q = G^{-1}$ denotes its inverse. The regular matrix Q determines the torus completely, $T = T(Q)$. The dual lattice is

$$\Gamma^* = \mathrm{span}_{\mathbb{Z}}\{v_1^*, \ldots, v_n^*\},$$

with $v_1^*, \ldots, v_n^*$ the dual basis, $v_i^*(v_j) = \delta_{ij}$.

The *Laplace–Beltrami operator* Δ on T takes the form

$$\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Its spectrum relative to the torus $T(Q)$ consists only of isolated eigenvalues with finite multiplicity and can be computed explicitly,

$$\mathrm{spec}_T(\Delta) = \{4\pi^2 q^\top Q q : q \in \mathbb{Z}^n\}.$$

A certain eigenvalue $\lambda \in \mathrm{spec}_T(\Delta)$ may correspond to multiple *representing vectors*, i.e. vectors $q \in \mathbb{Z}^n$ satisfying $\lambda = 4\pi^2 q^\top Q q$. The number of distinct representing vectors relative to the eigenvalue λ is called the *degeneracy* of λ. Thus the degeneracy of a given λ equals the dimension of its eigenspace, a basis of which is given by the functions

$$\Psi_q : T \ni x \longmapsto \exp\left(2\pi i \sum_{i=1}^{n} q_i v_i^*(x)\right) \in \mathbb{C}$$

where $q = [q_1, \ldots, q_n]^\top \in \mathbb{Z}^n$ is a representing vector of λ.

Let $f : M \to \mathbb{R}$ be a function on a compact manifold M. Then the *nodal domains* of f are defined as the connected components of $M \backslash f^{-1}(0)$, the number of which is finite. Throughout this work we will consider the nodal domains of the real and imaginary parts of the eigenfunctions Ψ_q. We will count these domains in the same way as introduced in [3], which goes as follows: first split Ψ_q into its real and imaginary parts,

$$\Psi_q^{\mathrm{re}}(x) = \cos\left(2\pi \sum_{i=1}^{n} q_i v_i^*(x)\right), \qquad \Psi_q^{\mathrm{im}}(x) = \sin\left(2\pi \sum_{i=1}^{n} q_i v_i^*(x)\right).$$

Then introduce the transformation

$$T(Q) \ni x \longmapsto Q^{-1}x = y \in \tilde{T}$$

onto the standard torus $\tilde{T} = \mathbb{R}^n/\mathbb{Z}^n$ which gives rise to the Laplace–Beltrami operator

$$\tilde{\Delta} = -\sum_{i,j} Q_{ij} \frac{\partial^2}{\partial y_i \partial y_j}$$

and to the functions

$$\tilde{\Psi}_q^{\mathrm{re}}(y) = \cos(2\pi q^\top y), \qquad \tilde{\Psi}_q^{\mathrm{im}}(y) = \sin(2\pi q^\top y).$$

The *number of nodal domains* is, by definition, the number given by lifting these functions to $\mathbb{R}^n$ and then counting the nodal domains in the unit cube ignoring identifications at the boundary. The resulting number, which we call the *nodal count* $\nu(q)$ for a given vector $q \in \mathbb{Z}^n$, is given by the following formula (see [3]):

$$\nu(q) = \begin{cases} 2\sum_{i=1}^{n} |q_i| & \text{for} \quad \Psi_q^{\mathrm{im}} \\ 2\sum_{i=1}^{n} |q_i| + 1 & \text{for} \quad \Psi_q^{\mathrm{re}}. \end{cases}$$

Since $T(Q)$ is a compact manifold we are able to arrange its spectrum $\mathrm{spec}_T(\Delta)$ in increasing order:

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots.$$

If we compute the nodal count of every vector $q \in \mathbb{Z}^n$, a finite set of nodal counts $\{\nu_1^i, \nu_2^i, \ldots\}$ belongs to each eigenvalue λ_i. The cardinality of this set equals the degeneracy of the corresponding eigenvalue. One obtains the *nodal sequence*

$$\left\{\{\nu_1^1, \nu_2^1, \ldots\}, \{\nu_1^2, \nu_2^2, \ldots\}, \ldots, \{\nu_1^i, \nu_2^i, \ldots\}, \ldots\right\}$$

by fitting each subsequence in the same position as the corresponding eigenvalue in the spectrum. By means of this sequence we shall distinguish isospectral tori.

3. The construction of Conway and Sloane

Our work deals with the 4-parameter family of isospectral flat tori $T^{\pm}(a, b, c, d)$ discovered by Conway and Sloane [2]. As mentioned in the previous section, these tori are described by the inverse $Q^{\pm}(a, b, c, d)$ of the corresponding Gram matrix. Explicitly,

$$Q^+ = \frac{1}{12}\begin{bmatrix} 9a+b+c+d & 3a-3b-c+d & 3a+b-3c-d & 3a-b+c-3d \\ 3a-3b-c+d & a+9b+c+d & a-3b+3c-d & a+3b-c-3d \\ 3a+b-3c-d & a-3b+3c-d & a+b+9c+d & a-b-3c+3d \\ 3a-b+c-3d & a+3b-c-3d & a-b-3c+3d & a+b+c+9d \end{bmatrix},$$

$$Q^- = U^\top Q^+ U \qquad \text{with} \qquad U = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \end{bmatrix}.$$

The defining parameters a, b, c, d are required to be strictly positive. It is remarked in [2] that the tori T^+ and T^- are equivalent, if two of these parameters are equal. Therefore we shall only consider vectors (a, b, c, d) of pairwise distinct positive numbers. We are now ready for the

Proof of the Theorem. To begin with we define for $m \in \mathbb{N}$ the set

$$V_m := \left\{ [q_1, q_2, q_3, q_4]^T \in \mathbb{Z}^4 : \sum_i |q_i| = m \right\}.$$

This set is obviously finite and contains all vectors $q \in \mathbb{Z}^4$ that represent the nodal count $2m$ or $2m+1$, according to whether we consider Ψ_q^{im} or Ψ_q^{re}. We then define

$$E_m^{\pm} := \{4\pi^2 q^\top Q^{\pm} q : q \in V_m\}$$

as the set of eigenvalues with a representing vector of nodal count $2m, 2m+1$. Thus, if E_m^+ and E_m^- do not coincide for a certain m, then the nodal sequences of the tori T^+ and T^- are distinct.

The $E_m^{\pm}$'s can be viewed as sets of functions in the variables a, b, c, d. By inspection we obtain equality $\left(E_m^+ = E_m^-\right)$ for $m = 1, 2, 3$. The first interesting case appears for $m = 4$, where

$$\begin{aligned}
E_4^+ = (4\pi^2/3)\{&(4a+25b+c), (25a+b+4c), (a+4b+25c),\\
&(b+25c+4d), (25a+4b+d), (25b+4c+d),\\
&(4a+25c+d), (4a+b+25d), (a+25b+4d),\\
&(25a+c+4d), (4b+c+25d), (a+4c+25d),\\
&(4a+16b+9c+d), (9a+4b+16c+d),\\
&(16a+9b+4c+d), (9a+16b+c+4d),\\
&(16a+b+9c+4d), (16a+4b+c+9d),\\
&(a+16b+4c+9d), (4a+b+16c+9d),\\
&(4a+9b+c+16d), (a+4b+9c+16d),\\
&(a+9b+16c+4d), (9a+b+4c+16d)\} \cup \left(E_4^+ \cap E_4^-\right),
\end{aligned}$$

$$\begin{aligned}
E_4^- = (4\pi^2/3)\{&(25a+4b+c), (a+25b+4c), (4a+b+25c),\\
&(4a+25b+d), (25a+4c+d), (4b+25c+d),\\
&(25a+b+4d), (25b+c+4d), (a+25c+4d),\\
&(a+4b+25d), (4a+c+25d), (b+4c+25d),\\
&(9a+16b+4c+d), (16a+4b+9c+d),\\
&(4a+9b+16c+d), (16a+9b+c+4d),\\
&(a+16b+9c+4d), (4a+16b+c+9d),\\
&(16a+b+4c+9d), (a+4b+16c+9d),\\
&(9a+4b+c+16d), (4a+b+9c+16d),\\
&(9a+b+16c+4d), (a+9b+4c+16d)\} \cup \left(E_4^+ \cap E_4^-\right).
\end{aligned}$$

Inspecting the sets E_4^+ and E_4^- more carefully one notes that there are two sets of coefficients, namely $(1, 4, 9, 16)$ and $(0, 1, 4, 25)$, such that E_4^+ contains all even permutations of the variables a, b, c, d in the linear forms with these coefficients while E_4^- contains the odd ones. Hence we may assume that $a < b < c < d$ and obtain a unique maximum among all elements of $E_4^+ \cup E_4^-$, namely $b + 4c + 25d$. Consequently $E_4^+ \neq E_4^-$, as claimed. □

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